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Complex Variable Method for Eigensolution Sensitivity Analysis

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The application of complex variable method for eigenvalue and eigenvector sensitivity analysis is presented. Gradient-based methods usually used in structural optimization require accurate sensitivity information. Complex variable method is accurate, robust, and easy to implement as compared to other approximate sensitivity analysis methods. By complex variable method, the first-order modal sensitivity can be calculated using an existing structural analysis package. This is achieved by making a small complex perturbation to a design variable. The resulting eigensolutions would be complex. The sensitivity of eigensolutions with respect to that design variable can then be obtained by taking the complex part of the eigensolutions divided by the perturbation. Because there is no subtraction involved in the sensitivity calculation, the approximation is not sensitive to step size. Thus, unlike finite difference methods, complex variable method does not suffer the effect of subtraction-cancellation error when a very small step size is used. The computation of first-order modal sensitivity is not hampered by repeated eigenvalue problem, as is the case with other sensitivity analysis methods. For eigenvalues of multiplicity greater than 1, complex variable method automatically provides correct eigenvalue sensitivity and the associated eigenvector sensitivity for a differentiable mode. A 5-degree-of-freedom system with an eigenvalue of multiplicity 3 is used to illustrate the salient features of complex variable method in eigensolution sensitivity analysis.

Nomenclature

K	=	stiffness matrix for spring-mass system
M	=	mass matrix for spring-mass system
λ	=	eigenvalue for spring-mass system
ϕ	=	eigenvector for spring-mass system

Superscript

$'$	=	derivative with respect to design variable

I. Introduction

EIGENSOLUTION sensitivity analysis is an important research area in structural optimization. Accurate sensitivity information helps designers to modify or redesign structures. Various analytical methods for computing first-order eigenvalue and eigenvector sensitivity are available and a survey was made by Adelman and Haftka [1]. Note that these methods are valid only for structures with distinct eigenvalues, because their eigenvectors are uniquely defined. But a designer can encounter repeated or near-repeated eigenvalues during the initial stages of design or in case of symmetric structures. In this case the eigenvectors are not uniquely defined and the previous sensitivity analysis methods cannot be used. There are some analytical methods for sensitivity analysis of eigenvalues of multiplicity greater than 1, as suggested by Haug and Rousselet [2], Choi and Haug [3], Ojalvo [4], Dailey [5], Mills-Curran [6,7], and Chen [8,9]. But none of these methods are general enough.

Analytical differentiation, finite differencing (FD), and automatic differentiation [10] (AD) are popular approaches to performing sensitivity analysis (SA). FD is easy to implement. However, the difficulties it presents are the computational expense, the step size selection, and errors due to subtraction-related cancellation. The user must balance roundoff error with truncation error to get decent sensitivities by FD. In recent years, with the increasing computing power, AD has drawn attention of researchers in SA. Based on chain rule of differential calculus, AD differentiates a source code of an analysis program and generates a program for sensitivity analysis. The use of AD is limited by the availability of source codes and the potential need to restructure portions of the code to make them amenable to AD. The use of complex variable method (CVM) for computing sensitivities in simulations is relatively new, compared to the other two approaches. In this work, the use of CVM for eigensolution sensitivity analysis will be discussed. We begin with an analytical formulation to calculate the modal sensitivity of a spring-mass system with distinct and repeated eigenvalues. Then, the CVM and its application to eigensolution sensitivity analysis are presented via a numerical example. Finally, conclusions are drawn from our present work.

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II. Analytical Formulation for Eigensolution Sensitivity

Consider a general eigenproblem:

$$K\phi = \lambda M\phi \tag{1}$$

Assuming Eq. (1) has nonrepeated roots, the first-order analytical eigensolution sensitivity [11] can be found by solving the system of linear equations shown in Eq. (2).

$$Az = b \tag{2a}$$

where

$$A = \begin{bmatrix} K - \lambda M & -M\phi \\ -(M\phi)^T & 0 \end{bmatrix} \qquad b = \begin{bmatrix} -(K' - \lambda M')\phi \\ 0.5\phi^T M'\phi \end{bmatrix} \qquad z = \begin{Bmatrix} \phi' \\ \lambda' \end{Bmatrix} \tag{2b}$$

$()' = \partial/\partial\alpha$, $\alpha =$ a design parameter. λ, ϕ are the solution of eigenvalue problem represented by Eq. (1), and ϕ is normalized so that $\phi^T M \phi = 1$. If Eq. (1) has repeated roots, the first-order analytical eigensolution sensitivity [8] can be computed as follows.

Let the three eigenvalues λ_i, λ_j, and λ_k be repeated and the eigenvectors corresponding to them be ϕ_i, ϕ_j, and ϕ_k, respectively. Then, the general procedure is to obtain the eigenvalue sensitivity followed by determining the corresponding unique eigenvector followed by eigenvector sensitivity. Let

$$\bar{\phi} = \begin{bmatrix} \phi_i & \phi_j & \phi_k \end{bmatrix} \tag{3}$$

The eigenproblem represented by Eq. (4) yields eigenvalues that correspond to the derivatives of repeated eigenvalues, and eigenvectors that can be used to generate unique eigenvectors associated with the repeated eigenvalues.

$$\{\bar{\phi}^T(K' - \lambda_i M')\bar{\phi} - \lambda_i'[I]\}\{a_i\} = \{0\} \tag{4}$$

In Eq. (4), $[I]$ is a unit matrix of dimension corresponding to the multiplicity of the repeated eigenvalues. Then, the unique eigenvector that corresponds to the repeated eigenvalue λ_i is given by Eq. (5):

$$\bar{\phi}_{i(\text{unique})} = \bar{\phi}\{a_i\} \tag{5}$$

The sensitivity of ϕ_i is then obtained by solving the system of linear equations shown in Eq. (6).

$$\bar{A}\phi_i' = \bar{b} \tag{6a}$$

where

$$\bar{A} = \begin{bmatrix} K - \lambda_i M \\ \bar{\phi}^T_{i(\text{unique})} M \\ \bar{\phi}^T_{j(\text{unique})} M \\ \bar{\phi}^T_{k(\text{unique})} M \end{bmatrix} \qquad \bar{b} = \begin{bmatrix} -(K' - \lambda_i' M - \lambda_i M')\bar{\phi}_{i(\text{unique})} \\ 0.5\bar{\phi}_{i(\text{unique})} M' \bar{\phi}^T_{i(\text{unique})} \\ \bar{\phi}_{j(\text{unique})} M' \bar{\phi}^T_{i(\text{unique})} \\ \bar{\phi}_{k(\text{unique})} M' \bar{\phi}^T_{i(\text{unique})} \end{bmatrix} \tag{6b}$$

The first n rows of matrix Eq. (6) are obtained by differentiation of the original eigenproblem, where n represents the degrees of freedom of spring-mass system under consideration. The $(n+1)$st row is obtained by mass-normalization of the unique eigenvector $\bar{\phi}_{i(\text{unique})}$. The $(n+2)$nd and $(n+3)$rd rows are obtained by M-orthogonality of the unique eigenvector $\bar{\phi}_{i(\text{unique})}$ with respect to $\bar{\phi}_{j(\text{unique})}$ and $\bar{\phi}_{k(\text{unique})}$, respectively. The sensitivity of $\bar{\phi}_j$ can be obtained similarly. This approach can be extended if the eigenvalue multiplicity is other than 3. For example, if a system has four repeated eigenvalues, then there would be an additional $(n+4)$th row corresponding to M- orthogonality of the unique eigenvector $\bar{\phi}_{i(\text{unique})}$ with respect to, say, $\bar{\phi}_{n(\text{unique})}$. In case the system has only two repeated eigenvalues, the $(n+3)$rd row in matrix Eq. (6) must be eliminated. Note that the M-orthogonality conditions can be combined together making use of matrix algebra although each M-orthogonality condition has been listed separately in matrix Eq. (6) for clarity. The reader is encouraged to refer to [8] for detailed derivation of analytical sensitivity analysis for repeated eigenvalues.

III. CVM for First-Order Sensitivity

CVM was developed by Lyness and Moler [12] and Lyness [13] in 1967 and a short summary was presented by Squire and Trapp [14].

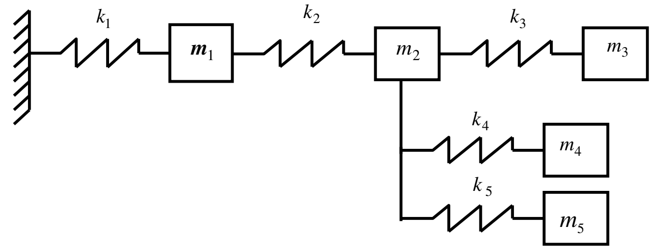

Fig. 1 Spring-mass system.

CVM has been applied to Navier–Stokes equation (Anderson et al. [15]) and has been studied extensively by Martins et al. [16]. Essentially, in CVM, we evaluate a real function at a point with a small imaginary perturbation ($i\varepsilon$) of a particular variable. The imaginary part of the function divided by the step size ε is an approximation to the derivative of the function with respect to the perturbed variable. Because no difference is taken, we may use a very small step size in CVM to get excellent approximation for the derivatives. Thus, CVM is very easy to implement, but it requires all computation to be performed in complex mode. Following is the derivation of CVM. Let $f(x)$ be an analytic function. The Taylor series expansion of $f(x + i\varepsilon)$ is

$$f(x + i\varepsilon) = f(x) + i\varepsilon f'(x) - \varepsilon^2 f''(x) - i\varepsilon^3 f'''(x) \ldots \tag{7}$$

Taking into account that $f(x)$ does not have an imaginary part, the preceding equation yields

$$f'(x) = \frac{\text{Im}[f(x + i\varepsilon)]}{(\varepsilon)} + \mathcal{O}(\varepsilon^2) \tag{8}$$

where $\text{Im}(f)$ is the imaginary part of f. Thus, we get the approximate formula for the derivative of $f(x)$ at x as shown in Eq. (9):

$$f'(x) \approx \frac{\text{Im}[f(x + i\varepsilon)]}{(\varepsilon)} \tag{9}$$

From Eqs. (8) and (9), we can say that this approximation has a truncation error associated with ε, which is of the order ε^2. Also, from Taylor's series expansion as shown in Eq. (7), it is observed that the function value at design x can be obtained by taking the real part of $f(x + i\varepsilon)$ with a truncation error of the order ε^2.

$$f(x) = \text{Re}[f(x + i\varepsilon)] + \mathcal{O}(\varepsilon^2) \tag{10}$$

where $\text{Re}(f)$ is the real part of f. Thus, we get the approximate formula for the function value $f(x)$ at x as shown in Eq. (11):

Table 1 Eigenvalues and eigenvectors of system distinct eigenvalues

Mode	1	2	3	4	5
λ_i	0.4605	6.9826	12.9628	24.2611	59.3329
ϕ_i'	0.1165	−0.0742	0.1259	−0.6614	0.1660
	0.1906	−0.0892	0.1010	−0.0326	−0.3800
	0.1999	−0.2956	−0.3409	0.0228	0.0770
	0.2100	0.2249	−0.0634	0.0085	0.0350
	0.1951	−0.1370	0.2871	0.1529	0.1932

Table 2 Eigenvalue and eigenvector sensitivities of system distinct eigenvalues with respect to k_2

Mode	1	2	3	4	5
λ_i'	5.4927e-3	2.2308e-4	6.1819e-3	0.3955	0.2982
ϕ_i'	1.4153e-3	4.1301e-4	−9.8574e-4	−1.7202e-3	6.4217e-3
	−1.9772e-4	−2.8728e-6	3.2559e-5	3.4392e-3	2.0362e-4
	−9.2189e-5	1.2331e-5	−3.8754e-5	−1.7781e-3	−5.0695e-4
	3.6306e-5	3.2552e-5	−1.5520e-5	−7.1915e-4	−2.1069e-4
	−1.4752e-4	−2.0656e-6	1.1775e-4	−1.9515e-3	−1.5686e-3

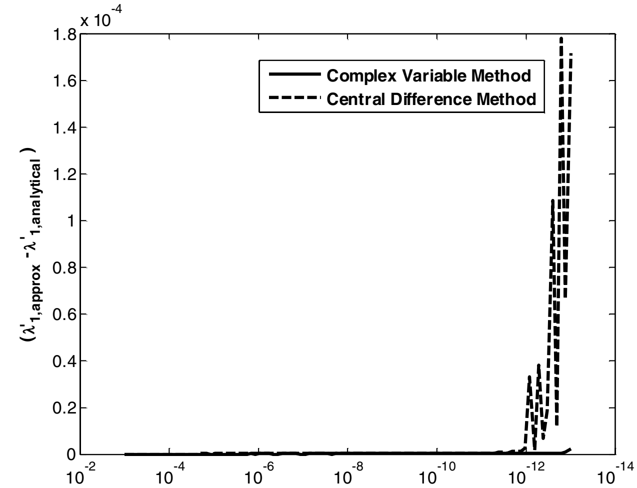

Fig. 2 $\sqrt{(\partial\lambda_1/\partial k_2)^2}$ vs step size.

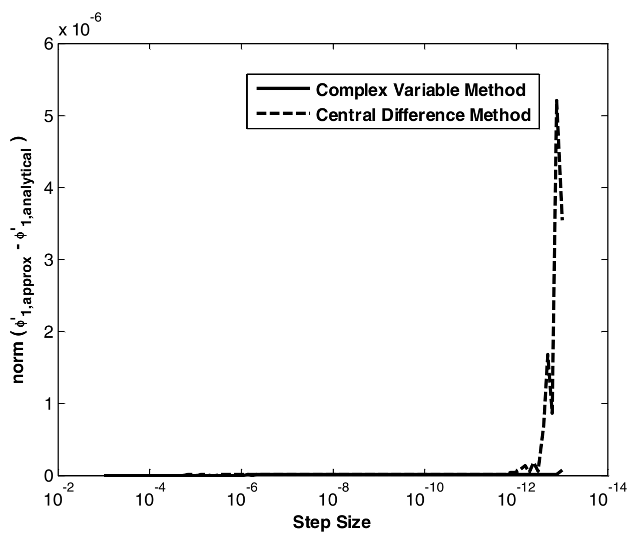

Fig. 3 $\sqrt{[\sum(\partial\phi_1/\partial k_2)]^2}$ vs step size.

$$f(x) \approx \mathrm{Re}[f(x+i\varepsilon)] \tag{11}$$

From Eqs. (9) and (11), it is observed that CVM yields function value as well as the sensitivity with only one function evaluation in complex design space, with a truncation error of the order ε^2.

IV. Numerical Example

Consider the spring-mass system shown in Fig. 1. The values of spring stiffness and masses $k_1 = 20$, $k_2 = 30$, $k_3 = 40$, $k_4 = 50$, $k_5 = 100$, $m_1 = 2$, $m_2 = 5$, $m_3 = 4$, $m_4 = 10$, and $m_5 = 5$ yield distinct eigenvalues and eigenvectors listed in Table 1. The eigenvalue and eigenvector sensitivities with respect to design variable k_2 are listed in Table 2.

Table 3 Eigenvalues and eigenvectors of system repeated eigenvalues

Mode	1	2	3	4	5
λ_i	0.2008	10.0000	10.0000	10.0000	49.7992
ϕ_i	−0.1014	−0.6067	0.1448	−0.3133	−0.0503
	−0.1988	0.0000	0.0000	0.0000	0.4006
	−0.2029	0.1819	−0.1918	−0.3590	−0.1007
	−0.2029	−0.0740	−0.1203	0.1696	−0.1007
	−0.2029	0.1239	0.3650	0.0106	−0.1007

Table 4 Eigenvalue and eigenvector sensitivity with respect to m_1

Mode	1	2	3	4	5
λ_i'	−0.0021	−4.8717	0.0000	0.0000	−0.1261
ϕ_i'	0.0507	−0.0051	0.0000	0.0000	0.0312
	−0.0021	−0.0320	0.0000	0.0000	0.0010
	−0.0021	−0.0102	0.0000	0.0000	−0.0006
	−0.0021	−0.0102	0.0000	0.0000	−0.0006
	−0.0021	−0.0102	0.0000	0.0000	−0.0006

Table 5 Eigenvalue and eigenvector sensitivity with respect to m_3

Mode	1	2	3	4	5
λ_i'	−0.0083	−1.9872	0.0000	0.0000	−0.5046
ϕ_i'	−0.0009	−0.0026	0.0000	0.0000	0.0012
	0.0082	0.0229	0.0000	0.0000	−0.0041
	−0.0029	−0.0053	0.0000	0.0000	−0.0292
	−0.0019	−0.0053	0.0000	0.0000	0.0023
	−0.0019	−0.0053	0.0000	0.0000	0.0023

Table 6 Eigenvalue and eigenvector sensitivity with respect to k_2

Mode	1	2	3	4	5
λ_i'	0.0095	0.4872	0.0000	0.0000	0.2033
ϕ_i'	0.0049	−0.0011	0.0000	0.0000	0.0054
	−0.0004	0.0018	0.0000	0.0000	−0.0002
	−0.0002	−0.0022	0.0000	0.0000	−0.0002
	−0.0002	−0.0022	0.0000	0.0000	−0.0002
	−0.0002	−0.0022	0.0000	0.0000	−0.0005

Table 7 Eigenvalue and eigenvector sensitivity with respect to k_5

Mode	1	2	3	4	5
λ_i'	1.6593*e*-5	0.1487	0.0000	0.0000	0.2513
ϕ_i'	−0.0859*e*-4	−0.0002	0.0000	0.0000	0.0003
	−0.1649*e*-4	0.0020	0.0000	0.0000	0.0000
	−0.1717*e*-4	−0.0005	0.0000	0.0000	−0.0006
	−0.1717*e*-4	−0.0005	0.0000	0.0000	0.0006
	0.6596*e*-4	−0.0005	0.0000	0.0000	−0.0019

Figures 2 and 3 show a comparison between CVM and CD for eigenvalue and eigenvector sensitivities for the first mode. Note that the CD diverges as step size decreases. This can be attributed to subtraction-cancellation error. Also, when the step size is further decreased below the value shown in Figs. 2 and 3, numerical noise produces large errors in both CVM and CD. The issue of numerical noise should be investigated later.

Note that using CVM, the modes must be mass-normalized in complex design space to obtain correct eigenvector sensitivity. Also, using an eigensolver like Matlab[‡] which produces modes sorted in ascending order of eigenvalues, the modes obtained in complex design space must be sorted again based on the real part of eigenvalues to obtain correct sensitivity.

Another important feature of CVM observed is that it provides correct sensitivity information for problems with repeated eigenvalues. The analytical sensitivity analysis of problems with repeated eigenvalue is not as straightforward as the one with distinct eigenvalues. It requires formulation of additional subproblems, which can be tedious. Using CVM the sensitivity of repeated eigenvalues can be obtained in the same way as distinct eigenvalues

[‡]Data available on-line at http://www.mathworks.com [cited 4 October 2006].

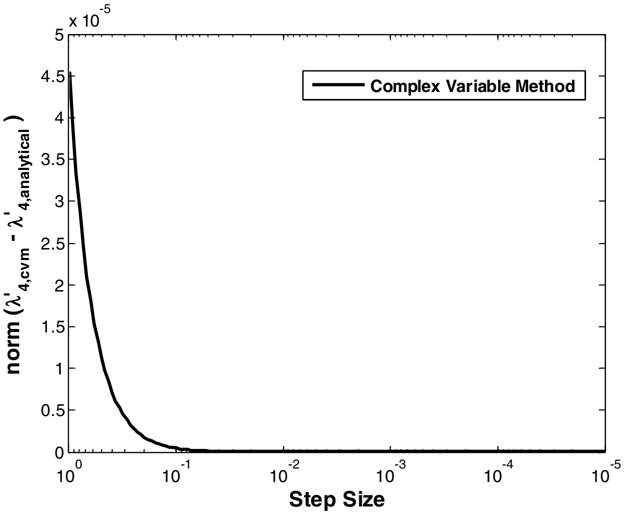

Fig. 4 $\sqrt{(\partial\lambda_4/\partial k_2)^2}$ **vs step size.**

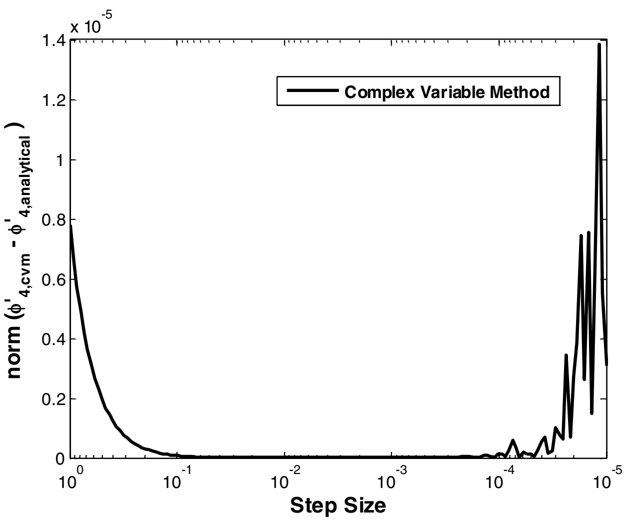

Fig. 5 $\sqrt{[\sum(\partial\phi_4/\partial k_2)]^2}$ **vs step size.**

without any additional formulations. The following example demonstrates the application of CVM in eigensensitivity of problems with repeated eigenvalues. The values of spring stiffness and masses $k_1 = 10$, $k_2 = 10$, $k_3 = 40$, $k_4 = 100$, $k_5 = 50$, $m_1 = 2$, $m_2 = 5$, $m_3 = 4$, $m_4 = 10$, $m_5 = 5$ produce three repeated eigenvalues. The eigenvalues and eigenvectors of the system are listed in Table 3. The sensitivities were computed with respect to masses (m_1 and m_3) and the spring stiffness (k_2 and k_5), which are considered to be design variables, using CVM and analytical method. The step size used in CVM was $1e$-3. Tables 4–7 list the respective eigenvalue and eigenvector sensitivities. The CVM and analytical sensitivities were in close agreement with the error norm between analytical and CVM of the order $1e$-10. To ensure that the truncation error is negligible for CVM, the step size should be at most the square root of "machine zero" which for 32-bit machines using double precision is $1e$-8. However, the numerical noise was observed when step size was reduced, as indicated in Figs. 4 and 5, and hence the step size of $1e$-3 was used in this example.

The effect of step size on eigenvalue and eigenvector sensitivities is demonstrated in Figs. 4 and 5, respectively. Note that there is only a small step size window ($1e$-1 to $1e$-5) for which CVM yields accurate sensitivity for the repeated fourth mode in this example. This can be attributed to the way Matlab handles computation in complex design space for small perturbations and the effect of numerical noise.

V. Conclusions

The CVM was successfully applied to eigensolution sensitivity analysis. CVM is accurate, robust, and easy to implement. Its application to systems with repeated eigenvalues is noteworthy. The examples presented clearly illustrate the ease with which CVM can be applied to general eigensolution sensitivity analysis problems. Moreover, CVM provides function value and the eigensensitivity of the perturbed variable with just one complex function evaluation. This helps to offset the additional time required for function evaluation in complex design space. It should be noted that CVM fails in case of structural systems that produce imaginary eigenvalues. But majority of structural systems can be modeled as producing real eigenvalues (e.g., undamped, nonrotating), thus making CVM useful for sensitivity analysis. Another issue in implementing CVM in commercial eigensolvers is that they must be able to solve eigenvalue problems in complex design space. This might require rewriting the source code to be compatible with complex algebra. Choosing an appropriate method for the sensitivity analysis involves a tradeoff among speed, ease of implementation, and stability. CVM is a useful addition to the set of SA methods.

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